SOLUTION TO CERTAIN CONTACT PROBLEMS WITHOUT AXIAL SYMMETRY IN THE CASE OF A PUNCH WITH AN ELLIPTICAL CROSS SECTION

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A solution is obtained to two new contact problems in the class of problems considered earlier [3]. The analytically established relations are suitable for use in several mechanical and engineering applications as, for instance, in the problem of heat and electric current conduction through a contact between solid deformable bodies. The most essential parameters have been computed and the results are shown here in the form of graphs.

The method of solution and the results shown in [1, 2] have made it feasible to obtain a general solution to a new class of problems concerning the penetration into an elastic half-space $Z > 0$ of an asymmetric smooth and rigid punch whose sections in the Z = const. planes are confocal ellipses:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1.$$

Here a is the major semiaxis, $b = \sqrt{a^2 - l^2}$ is the minor semiaxis, $2l$ is the focal distance, and X, Y, Z are Cartesian coordinates related to the ellipsoidal coordinates which will be used here as follows:

$$X = r\cos\theta, \quad Y = \sqrt{r^2 - l^2}\,\sin\theta, \qquad \frac{D(X,\ Y)}{D(r,\ \theta)} = \frac{r^2 - l^2\cos^2\theta}{\sqrt{r^2 - l^2}}.$$

The punch generatrix can be specified by any function $Z = f(a)$. Correspondingly, the punch surface will be described by the equation

$$Z = f[a(X,\ Y)].$$

The expressions for the pressure p_0 under a flat elliptical punch and for the displacement w_0 outside it [4]

$$p_0(X,\ Y;\ a) = \frac{A}{K\left(\frac{l}{a}\right)\sqrt{1 - \frac{l^2}{a^2}}\sqrt{a^2 - X^2 - \frac{Y^2}{1 - \frac{l^2}{a^2}}}}, \quad A = \frac{G}{1 - \nu},$$

$$w_0(X,\ Y;\ a) = w_0(r,\ \theta;\ a) = \frac{1}{K\left(\frac{l}{a}\right)}\, F\left(\arcsin\frac{a}{r},\ \frac{l}{a}\right),$$

together with the formulas given in [2] and [3] for the punch penetration α, the total contact force P or Q of a flat or deep punch respectively, for the pressure p under a punch, and for the displacement w outside it

$$\alpha(a) = \frac{N'(a)}{Q'(a)}, \quad P(a) = \alpha(a)\,Q(a) - N(a), \tag{1}$$

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$$\left.\begin{aligned}
Q(a) &= \int\limits_{S_a} p_0(X, Y; a)\, dXdY = \frac{2\pi A a}{K\left(\frac{l}{a}\right)}, \\
N(a) &= \int\limits_{S_a} p_0(X, Y; a) f(X, Y)\, dXdY \\
&= 4Aa^2 \frac{E\left(\frac{l}{a}\right)}{K\left(\frac{l}{a}\right)} \int\limits_l^a \frac{f(r)\, dr}{\sqrt{(a^2 - r^2)(r^2 - l^2)}} - 4A \int\limits_l^a \sqrt{\frac{a^2 - r^2}{r^2 - l^2}}\, f(r)\, dr,
\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
p(r, \theta; a) &= \int\limits_r^a p_0(r, \theta; t) \frac{d\alpha(t)}{dt}\, dt \\
&= A \int\limits_r^a \frac{\frac{d\alpha(t)}{dt}\, t dt}{K\left(\frac{l}{t}\right)\sqrt{(t^2 - r^2)(t^2 - l^2\cos^2\theta)}}, \\
w(r, \theta; a) &= \int\limits_l^a w_0(r, \theta; t) \frac{d\alpha(t)}{dt}\, dt \\
&= \int\limits_l^a \frac{\frac{d\alpha(t)}{dt}}{K\left(\frac{l}{t}\right)} F\left(\arcsin\frac{t}{r}, \frac{l}{t}\right) dt
\end{aligned}\right\} \tag{3}$$

(K, E, F denote two complete and one incomplete elliptic integrals) make it possible to write the exact solution for each specific function $f(a)$ in quadratures. Of most practical interest is the case of a narrow contact zone. In [3], for example, was examined with the generatrix $f = c(a-l)$ and the initial tangent to the solid surface along the straight segment $-l \le X \le l$, $Y = 0$. At X = const. sections its surface runs into the X axis along a parabola. When the major semiaxis a of the contact ellipse approaches the length l, then the contact zone becomes narrow and elongated; when a is much longer than l, on the other hand, then the problem approaches the case of a contact between an axially symmetric cone and an elastic half-space. In this study here will be considered two punches differing from the one in [3] by respectively opposite characteristics: one blunter and one sharper.

1. Let

$$f = c\,[a(X, Y) - l]^2, \quad a \geqslant l. \tag{4}$$

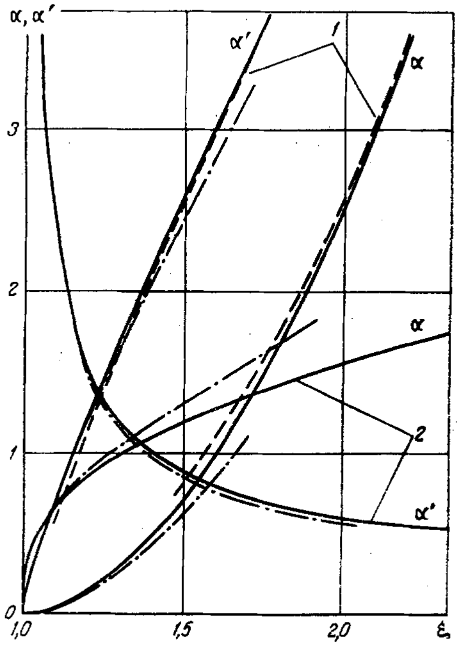
For the punch section in the Y = 0 plane we have

$$f[a(X, 0)] = \begin{cases} 0 & \text{for } |X| \leqslant l, \\ c(|X| - l)^2 & \text{for } |X| \geqslant l, \end{cases}$$

and at punch sections in X = const. planes the punch surface runs into the X axis along the $-l \le X \le l$ segment as a fourth-degree parabola. Thus, the punch is very rounded but elongated at small values of X and Y, becoming a paraboloid of revolution at large values of X and Y. The contact begins at the tangency along the segment $-l \le X \le l$, $Y = 0$.

In the dimensionless coordinates

$$\xi = \frac{a}{l}, \quad \eta = \frac{b}{l}, \quad \rho = \frac{r}{l}, \quad x = \frac{X}{l}, \quad y = \frac{Y}{l}, \quad \alpha(\xi) = \frac{\alpha(a)}{cl}, \tag{5}$$



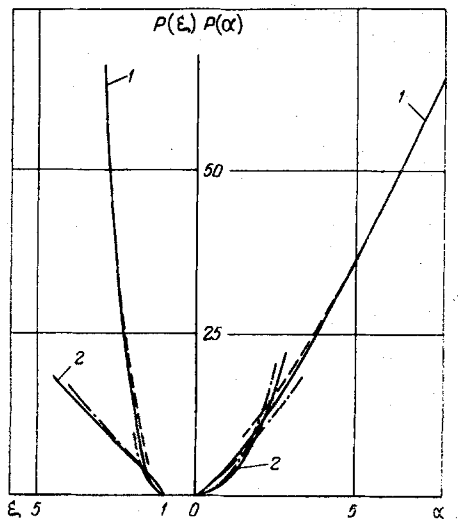


Fig. 1 Fig. 2

Fig. 1. Punch penetration α and its derivative α' as functions of the major semiaxis ξ of the contact ellipse. Dashed curves represent the asymptotics for large values ξ, dashed-dotted curves represent the asymptotics for small values of ξ: $f = c(a-l)^2$ (1), $f = c\sqrt{a-l}$ (2). All quantities are dimensionless.

Fig. 2. Total contact force P as a function of the penetration α and as a function of the major semiaxis of the contact ellipse ξ. Dashed curves represent the asymptotics for large values of ξ and α respectively, dashed-dotted curves represent the asymptotics for small values of ξ and α respectively: $f = c(a-l)^2$ (1), $f = c\sqrt{a-l}$ (2). All quantities are dimensionless.

$$f(\rho) = \frac{f(r)}{cl}, \quad P(\xi) = \frac{P(a)}{Acl^2}, \quad N(\xi) = \frac{N(a)}{Acl^2},$$
$$Q(\xi) = \frac{Q(a)}{Al}, \quad p(\rho, \theta; \xi) = \frac{p(r, \theta; a)}{Ac}. \tag{5}$$

Inserting (4) into (2) yields

$$Q(\xi) = \frac{2\pi\xi}{K_1},$$
$$N(\xi) = 4\xi K_2\left(\delta - \frac{4}{3}\right) + 2\pi\left[\xi^2(1-2\delta) - 1\right] + \frac{4}{3}\xi E_2\left[\xi^2(3\delta - 1) + 5\right].$$

The solution to the problem becomes

$$\alpha(\xi) = \frac{2}{\pi} K_1\{-\pi\delta\xi + K_2(\delta - 2) + E_2[\xi^2(1+\delta) + 1]\},$$
$$P(\xi) = \alpha(\xi)\,Q(\xi) - N(\xi), \tag{6}$$

$$p(\rho, \theta; \xi) = \int_{\rho}^{\xi} \frac{\dfrac{d\alpha(\tau)}{d\tau}\,\tau\,d\tau}{K\left(\dfrac{1}{\tau}\right)\sqrt{(\tau^2-\rho^2)(\tau^2-\cos^2\theta)}}, \tag{7}$$

$$w(\rho, \theta; \xi) = \int_{1}^{\xi} \frac{\dfrac{d\alpha(\tau)}{d\tau}}{K\left(\dfrac{1}{\tau}\right)} F\left(\arcsin\frac{\tau}{\rho}, \frac{1}{\tau}\right) d\tau, \tag{8}$$

where

$$\frac{d\alpha(\xi)}{d\xi} = \frac{2}{\pi} K_1 \left(-\pi - \frac{2}{\xi} K_2 + 4\xi E_2 \right);$$

$$K_1 = K\left(\frac{1}{\xi}\right); \quad E_1 = E\left(\frac{1}{\xi}\right); \quad \delta = \frac{E_1}{K_1}; \quad K_2 = K\left(\frac{\sqrt{\xi^2-1}}{\xi}\right); \tag{9}$$

$$E_2 = E\left(\frac{\sqrt{\xi^2-1}}{\xi}\right);$$

K and E are complete elliptic integrals.

According to formulas (6)-(9), the respective quantities were computed on a BÉSM-6 machine. Furthermore, asymptotic formulas have been derived for ξ values close to unity and much greater than unity.

In the first case ($\xi \sim 1$ or $0 < \xi - 1 \ll 1$) the zone is narrow and the asymptotic formulas are

$$\alpha(\xi) = \frac{3}{4}(\xi-1)^2 \ln\frac{8}{\xi-1} + \frac{3}{8}(\xi-1)^2 + O[(\xi-1)^2],$$

$$\alpha'(\xi) = \frac{3}{2}(\xi-1)\ln\frac{8}{\xi-1} + \frac{9}{16}(\xi-1)^2 \ln\frac{8}{\xi-1} + O\left[(\xi-1)^2 \ln\frac{8}{\xi-1}\right],$$

$$P(\xi) = 3\pi(\xi-1)^2\left[1 + \frac{1}{2\ln\left(\frac{8}{\xi-1}\right)} - \frac{1}{2\ln^2\left(\frac{8}{\xi-1}\right)}\right] + O\left[\frac{(\xi-1)^2}{\ln^2\left(\frac{8}{\xi-1}\right)}\right],$$

$$\begin{aligned} p(\rho, \theta; \xi) \approx{}& \frac{15}{4}\xi\sqrt{\frac{\xi^2-\rho^2}{\xi^2-\cos^2\theta}} - \frac{3}{16}\sqrt{(\xi^2-\rho^2)(\xi^2-\cos^2\theta)} \\ &+ \frac{15}{4}\rho\left[F\left(\arcsin\sqrt{\frac{\xi^2-\rho^2}{\xi^2-\cos^2\theta}}, \frac{\cos\theta}{\rho}\right)\right. \\ &\left. - E\left(\arcsin\sqrt{\frac{\xi^2-\rho^2}{\xi^2-\cos^2\theta}}, \frac{\cos\theta}{\rho}\right)\right] - \frac{3}{16}(18+\rho^2 \\ &+ \cos^2\theta)\ln\frac{\sqrt{\xi^2-\rho^2}+\sqrt{\xi^2-\cos^2\theta}}{\sqrt{\rho^2-\cos^2\theta}}, \end{aligned} \tag{10}$$

F and E are incomplete elliptic integrals. On the y-axis Eq. (10) simplifies to

$$p(0, y; \xi) \approx \frac{3}{16}\left[\sqrt{\eta^2-y^2}(20-\xi)\right] - \frac{3}{16}(19+y^2)\ln\frac{\sqrt{\eta^2-y^2}+\xi}{\sqrt{y^2+1}}.$$

Comparing these expressions with the exact solution (6)-(9), we ascertain that they are operative with a smaller than 5% error for α in the $1 < \xi < 1.2$ region, for α' in the $1 < \xi < 1.5$ region, for P in the $1 < \xi < 1.3$ region, and for p in the $1 < \xi < 2$ region (Figs. 1 and 2).

In the second case, when $\xi \gg 1$ and the contact zone approaches a circle, the asymptotic formulas

$$\alpha(\xi) = 2\xi^2 - \pi\xi + \frac{1}{2} + \frac{\pi}{4\xi} - \frac{1}{8\xi^2}\ln 4\xi - \frac{1}{16\xi^2} + \frac{3\pi}{64\xi^3} - \frac{1}{16\xi^4}\ln 4\xi + \frac{1}{64\xi^4} + \frac{\pi}{256\xi^5} + O\left(\frac{1}{\xi^6}\right),$$

$$\alpha'(\xi) = 4\xi - \pi - \frac{\pi}{4\xi^2} + \frac{1}{4\xi^3}\ln 4\xi - \frac{9\pi}{64\xi^4} + O\left(\frac{1}{\xi^4}\right),$$

$$P(\xi) = \frac{16}{3}\xi^3 - 2\pi\xi^2 - 4\xi + 2\pi - \frac{1}{4\xi}\ln 4\xi + \frac{1}{4\xi} - \frac{\pi}{2\xi^2} + O\left(\frac{1}{\xi^2}\right),$$

$$p(\rho, \theta; \xi) \approx \frac{2}{\pi}\left\{4\xi\sqrt{\frac{\xi^2-\rho^2}{\xi^2-\cos^2\theta}} - \pi\ln\frac{\sqrt{\xi^2-\rho^2}+\sqrt{\xi^2-\cos^2\theta}}{\sqrt{\rho^2-\cos^2\theta}}\right.$$

$$+ \left(4\rho - \frac{1}{\rho}\right)F\left(\arcsin\sqrt{\frac{\xi^2-\rho^2}{\xi^2-\cos^2\theta}}, \frac{\cos\theta}{\rho}\right)$$

$$- 4\rho E \left(\arcsin \sqrt{\frac{\xi^2 - \rho^2}{\xi^2 - \cos^2\theta}}, \ \frac{\cos\theta}{\rho} \right) \Bigg\} \tag{11}$$

are operative with a smaller than 5% error for α' in the $\xi > 1.25$ region and for P in the $\xi > 1.7$ region. With the first four terms of the asymptotic for α (represented by the dashed curve in Fig. 2) alone, the accuracy remains within 10% in the $\xi > 1.6$ region; with all ten terms the values of α remain accurate within 4% already when $\xi > 1.1$ and almost fall on the exact curve. The asymptotic of the contact pressure is accurate within 3% already when $\xi \geq 3$. Evidently, the asymptotics for $\xi \sim 1$ and $\xi \gg 1$ converge and sometimes merge without ceasing to be fairly accurate.

When $\xi \gg 1$, the first term of the α, α', and P asymptotics concur exactly with the solution for a paraboloid of revolution [5], the contact pressure for which is given by the expression

$$p(\rho;\ \xi) = \frac{8}{\pi}\sqrt{\xi^2 - \rho^2}.$$

The first term of the pressure asymptotic approaches this expression when $\xi \to \infty$, which is especially evident as expression (11) becomes

$$p(0,\ y;\ \xi) \approx \frac{2}{\pi}\left\{4\sqrt{\eta^2 - y^2} - \pi \ln \frac{\sqrt{\eta^2 - y^2} + \xi}{\sqrt{y^2+1}} - \frac{1}{\sqrt{y^2+1}} \operatorname{arctg} \frac{\sqrt{\eta^2 - y^2}}{\sqrt{y^2+1}}\right\}$$

on the y-axis.

As an application example, a numerical method has been developed in [1] for calculating the penetration α of an elliptical punch with the generatrix (4) as a function of the major semiaxis of the contact ellipse ξ. A comparison with the exact expression (6) reveals a complete concurrence of results (within computation accuracy).

Unlike in the case in [3], the pressure p is everywhere bounded: there are no singularities at points $x = \pm 1$ on the x-axis, because the punch corners are rounded.

2. We now consider a punch which is sharper than the one described in [3]. Its shape can be defined by the function

$$f = c\sqrt{a(X,\ Y) - l}, \quad a \geqslant l,$$

which at the punch section in the Y = 0 plane is

$$f[a(X,\ 0)] = \begin{cases} 0 & \text{for } |X| \leqslant l, \\ c\sqrt{|X| - l} & \text{for } |X| \geqslant l. \end{cases}$$

In X = const. sections on the interval $-l < X < l$ this punch runs into X axis linearly:

$$f \approx \frac{cY}{(1 - X^2)\sqrt{2 + X^2}},$$

i.e., the lower end of the punch is almost a wedge.

In the dimensionless coordinates (5) we have

$$\left.\begin{aligned} Q(\xi) &= \frac{2\pi\xi}{K_1}, \\ N(\xi) &= \frac{8\xi^2}{\sqrt{2\xi}}\delta F_3 - \frac{8}{3}\sqrt{2\xi}\,[2E_3 + (\xi - 1)F_3] + \frac{8}{3}\sqrt{2}\sqrt{\xi^2 - 1}, \end{aligned}\right\} \tag{12}$$

$$\alpha(\xi) = \frac{\sqrt{2}}{\pi}\frac{K_1}{\sqrt{\xi}}\left\{2\sqrt{\frac{\xi^2 - 1}{\xi}} + F_3(2\delta\xi - \xi + 1) - 2E_3\right\}, \tag{13}$$

$$\alpha'(\xi) = \frac{K_1}{\pi}\left[\frac{F_3}{\sqrt{2\xi}} + \sqrt{\frac{2}{\xi^2 - 1}}\right]. \tag{14}$$

Here

$$F_3 = F\left(\arccos\sqrt{\frac{2}{\xi + 1}},\ \sqrt{\frac{\xi + 1}{2\xi}}\right);\ E_3 = E\left(\arccos\sqrt{\frac{2}{\xi + 1}},\ \sqrt{\frac{\xi + 1}{2\xi}}\right);$$

F and E are incomplete elliptic integrals.

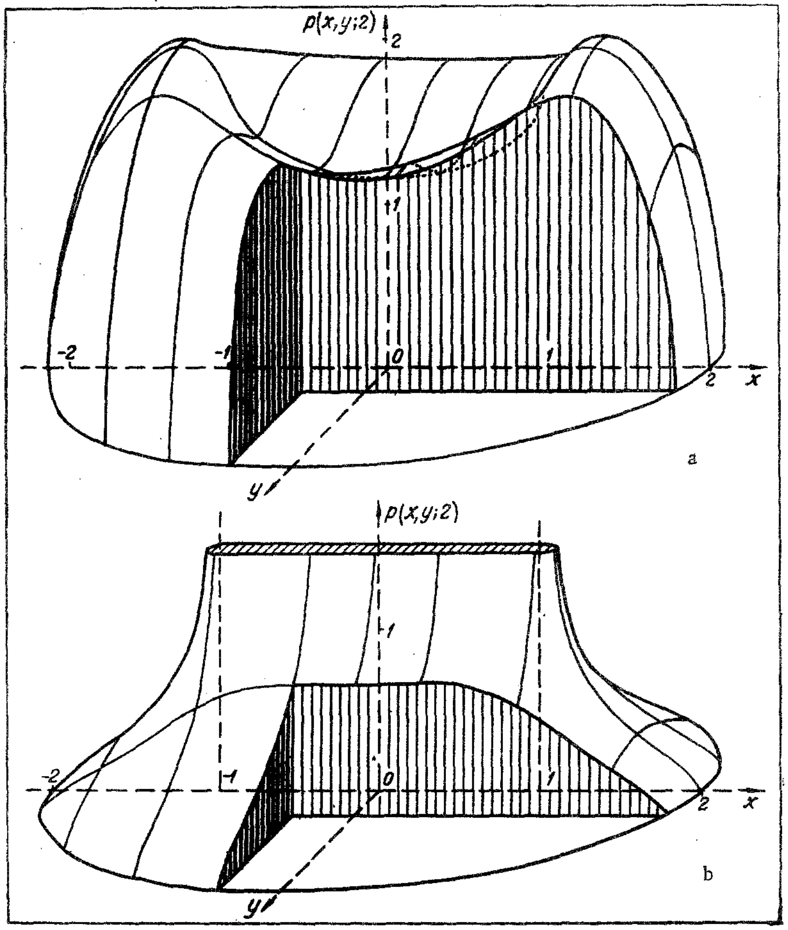


Fig. 3. Distribution of contact pressure p at $\xi = 2$ for cases a) $f = c(a = l)^2$ and b) $f = c\sqrt{a - l}$.

The quantities α, α', P, and p were computed on a BÉSM-6 machine according to formulas (7), (8), (12, (13), and (14).

The asymptotics were computed only for $\xi \sim 1$, inasmuch as for larger ξ values they become more unwieldy and also much less important:

$$\alpha(\xi) = \frac{\sqrt{\xi - 1}}{\pi} \ln \frac{8}{\xi - 1} + \frac{2\sqrt{\xi - 1}}{\pi} + \frac{2(\xi - 1)^{3/2}}{\pi} + O[(\xi - 1)^{3/2}],$$

$$\alpha'(\xi) = \frac{1}{2\pi\sqrt{\xi - 1}} \ln \frac{8}{\xi - 1} + \frac{3\sqrt{\xi - 1}}{8} \ln \frac{8}{\xi - 1}$$

$$+ O\left[\sqrt{\xi - 1} \ln \frac{8}{\xi - 1}\right],$$

$$P(\xi) = 4\sqrt{\xi - 1} + 2(\xi - 1)^{3/2} + O[(\xi - 1)^{3/2}].$$

These asymptotic formulas are accurate within 5% for α in the $1 < \xi < 1.3$ region, for α' in the $1 < \xi < 2$ region, and for P in the $1 < \xi < 2$ region. The principal part of the contact-pressure integral p with $\xi \sim 1$ is given by the expression

$$\int_r^{\xi} \frac{d\tau}{\sqrt{(\tau - 1)(\tau^2 - r^2)(\tau^2 - \cos^2\theta)}},$$

which indicates that the pressure is unbounded on the $-1 \le x \le 1$ segment of the x-axis ($r = 1$, $x = \cos\theta$). Some results of computations on the BÉSM-6 machine are shown in Fig. 3a, b. It is to be noted that the programs written in the FORTRAN language yield the contact pressure at any point within the contact region with any value of ξ. Computations similar to these can be made also for other functions f, but for the time being we have thoroughly analyzed three typical among the class of contact problems in [3].

NOTATION

r	is the radius;
ρ	is the dimensionless radius;
θ	is the angle, in elliptical coordinates;
X, Y, Z	are the Cartesian coordinates of a point;
x, y	are the dimensionless Cartesian coordinates of a point;
w	is the displacement of elastic material along a punch of arbitrary shape;
p	is the pressure under a punch of arbitrary shape;
w_0	is the displacement of elastic material along a flat punch;
p_0	is the pressure under a flat punch;
f	is the function which defines the punch shape;
P	is the total contact force of a punch of arbitrary shape;
Q	is the total contact force of a flat punch;
N	is the component of total force;
α	is the depth of punch penetration;
S_a	is the contact zone;
K, E, F	are the two complete and one incomplete elliptic integrals;
a, ξ	are the major semiaxis and dimensionless major semiaxis of an ellipse;
b, η	are the minor semiaxis and dimensionless minor semiaxis of an ellipse;
l	is one half the focal distance of an ellipse;
D	is the differentiation operator in a determinant;
A, c	are the coefficients in the pressure equation for a flat punch and in function f;
G	is the shear modulus for the half-space material.
ν	is the Poisson ratio for the half-space material.

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